A WEIGHT SYSTEM DERIVED FROM THE MULTIVARIABLE CONWAY POTENTIAL FUNCTION

HITOSHI MURAKAMI

Abstract. A weight system is defined from the (multivariable) Conway potential function. We also show that it can be calculated recursively by using five axioms.

1. Introduction

In [3] D. Bar-Natan and S. Garoufalidis used a weight system for the Alexander polynomial of knots to prove the so-called Melvin–Morton–Rozansky conjecture [18, 21] which relates the Alexander polynomial and some coefficients of coloured Jones polynomial. Their weight system can be easily extended to the case of links. It is a natural problem to give a weight system for the multivariable Alexander polynomial or its normalised version, the Conway potential function for links.

The Conway potential function was first introduced by J.H. Conway [6] by giving some 'axioms'. Unfortunately, his 'axioms' are not sufficient to define his potential function. R. Hartley [11] gave its precise definition by using R.H. Fox's free differential calculus [9, 10]. He also showed that for two-bridge links Conway's first two identities and initial data for the trivial knot, for split links, and for the positive Hopf link are sufficient to calculate the potential function. After that M.E. Kidwell [13] proved they are also sufficient for calculation of links with two labels $K = T \cup L$ where T is an unknotted circle and T and L have different labels. (Note that L may have more than one component.) Then Y. Nakanishi [20] proved that we can calculate the potential function if the number of labels (which equals the number of variables) is two or three. Besides Hartley's axioms we need Conway's third identity and initial data for the connected sum of two positive Hopf links and for the three-component positive Hopf link. Finally J. Murakami proved that Conway's first and second identities, a connect sum formula for the positive Hopf link, initial data for the trivial knot and for split links, and his new relation involving seven locally distinct links are sufficient to calculate the Conway potential function for links with any number of labels.

In this paper we use J. Murakami's result to define a weight system. Moreover we will show that our weight system can be calculated recursively by using five axioms. Since the proof is fairly easy we expect that there may be similar weight systems. If so, by using M. Kontsevich's integral [14] we could then define invariants for labelled links other than the Conway potential function. It is also an interesting problem whether our weight system is canonical, that is, whether we obtain the

1991 *Mathematics Subject Classification.* 57M25, 57M05.

The author is an EPSRC Visiting Fellow (grant number GR/K/46743) in the University of Liverpool.

To appear in J. London Math. Soc.

Conway potential function again after applying the composition of the Kontsevich integral and our weight system to links.

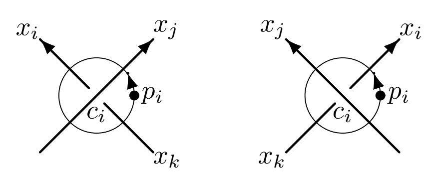
Acknowledgements. Most of the results here were prepared for the Knot Seminar at the University of Liverpool in the autumn term, 1995. The author thanks H.R. Morton for inviting him to Liverpool as an EPSRC fellow, for arranging the Seminar, and for valuable comments. He also thanks the audience of the Seminar for their patience with his poor *Engrish.* Thanks are also due to Y. Nakanishi for helpful conversations.

2. Preliminaries

In this section we define the Conway potential function and describe the concept of Vassiliev invariants.

In [6] J.H. Conway introduced a notion of the potential function by an axiomatic way. R. Hartley [11] gave its precise definition by using R.H. Fox's free differential calculus [9, 10] and proved its existence explicitly. We follow R. Hartley to give the definition of the Conway potential function.

Let $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be an oriented μ-component link with labels $1, 2, \ldots, n$. Here K_i is labeled with $\ell(i) \in \{1, 2, \ldots, n\}$ $(i = 1, 2, \ldots, \mu)$. Let $\mathcal{L}$ be a *connected* link diagram of L. Let $c_1, c_2, \ldots, c_m$ be the crossings and x_i the arc starting at c_i $(i = 1, 2, \ldots, m)$. We read the Wirtiger relation r_i at c_i anticlockwise starting at a point p_i to the right of both arcs. Then it is of the form $r_i = x_j x_i x_j^{-1} x_k^{-1}$ or $r_i = x_i x_j x_k^{-1} x_j^{-1}$ according as c_i is a positive crossing or a negative one. See Figure 1.



positive crossing negative crossing

Figure 1

Let $\varphi : \mathbb{Z}F(x_1, x_2, \ldots, x_m) \to \mathbb{Z}[{t_1}^{\pm 1}, {t_2}^{\pm 1}, \ldots, {t_n}^{\pm 1}]$ be the abelianisation homomorphism sending x_i to $t_{\ell(j)}$ if x_i belongs to K_j, where $\mathbb{Z}F(x_1, x_2, \ldots, x_m)$ is the group ring of the free group generated by m letters $x_1, x_2, \ldots, x_m$. We consider the $m \times m$ Jacobian matrix $M(\mathcal{L}) = \varphi(\partial r_i / \partial x_j)$, where $\partial/\partial x_j$ is Fox's free differential calculus [9, 10]. Let $D^{(ij)}(\mathcal{L})$ be the determinant of the matrix obtained from $M(\mathcal{L})$ by deleting the i-th row and the j-th column. Then $D^{(ij)}(\mathcal{L})/(\varphi(x_j) - 1)$ is the multivariable Alexander polynomial of the labelled link L (if $\mu > 1$; we do not need to divide by $\varphi(x_j) - 1$ if $\mu = 1$) and so defines the Conway potential function up to units of $\mathbb{Z}[{t_1}^{\pm 1}, {t_2}^{\pm 1}, \ldots, {t_n}^{\pm 1}]$.

To define the Conway potential function precisely we need more definitions. Let w_i be the word read from a path connecting a point in the unbounded region in $\mathbb{R}^2$ and p_i. Let $\kappa_g(\mathcal{L})$ be the rotation number (or curvature) of the sublink consisting of

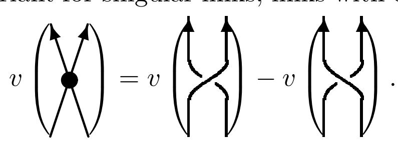
all the components labelled with g, which counts (algebraically) how many times the sublink rotates anticlockwise. Let $\nu_g(\mathcal{L})$ be the (geometric) number of the crossings where components labelled with g cross over. Then according to R. Hartley we can define the Conway potential function $\nabla_n(L; t_1, t_2, \dots, t_n)$ as

$$\nabla_n(L; t_1, t_2, \dots, t_n) = \frac{(-1)^{i+j} D^{(ij)}(\mathcal{L})}{\varphi(w_i)(\varphi(x_j) - 1)} \prod_{g=1}^{n} t_g{}^{(\kappa_g(\mathcal{L}) - \nu_g(\mathcal{L}))/2}.$$

Note that our definition differs from Conway's and Hartley's. Their potential function is $\nabla_n(L; t_1{}^2, t_2{}^2, \dots, t_n{}^2)$ in our notation.

It is well known [10] that $\nabla_n(L; t_1, t_2, \dots, t_n) \in \mathbb{Z}[t_1{}^{\pm 1/2}, t_2{}^{\pm 1/2}, \dots, t_n{}^{\pm 1/2}]$ if $\mu > 1$ and $(t_1{}^{1/2} - t_1{}^{-1/2})\nabla_1(L; t_1) \in \mathbb{Z}[t_1{}^{\pm 1/2}]$ if $\mu = 1$. In this paper, we study the Laurent expansion of $\nabla_n(L; \exp(h_1), \exp(h_2), \dots, \exp(h_n))$ at $(h_1, h_2, \dots, h_n) = (0, 0, \dots, 0)$ and denote it simply by $\nabla_n(L)$. So if $\mu > 1$, then $\nabla_n(L)$ is a Taylor series and if $\mu = 1$, then it is a Laurent series of the form $\sum_{k=-1}^{\infty} c_k h_1{}^k$.

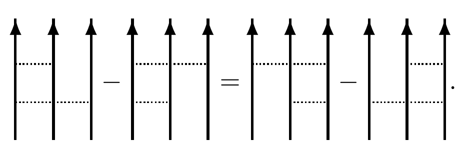
Next we describe Vassiliev invariants. Given a numerical link invariant v, we can also regard it as an invariant for singular links, links with double points, as follows.



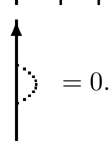
Now v is called a Vassiliev invariant of type d [22, 5, 4] if it vanishes for all the singular links with more than d double points. This is equivalent to saying that for any singular link L^d with d double points, $v(L^d)$ does not depend on its embedding; it depends only on the configuration how double points are paired on the circles [22, 5, 4]. Such a configuration is described by a chord diagram.

For a compact one-manifold N, $N \cup I_1 \cup I_2 \cup \dots \cup I_m$ is called *a chord diagram with support* N, where I_i is an interval $[0, 1]$ $(1 \le i \le m)$, $I_i \cap I_j = \emptyset$ $(i \ne j)$, and $N \cap I_i = N \cap \partial I_i = \partial I_i$ $(1 \le i \le m)$. We call a (part of) connected component of N an arc and I_i a chord. We use solid lines for arcs and dotted lines for chords. We denote by $\mathcal{D}(N)$ the set of linear combinations of chord diagrams with support N over $\mathbb{C}$. We denote $\mathcal{D}(N)$ modulo the following 4-term relation and the framing independence relation by $\mathcal{A}(N)$.

(4-term relation)



(framing independence relation)



M. Kontsevich defined by using the iterated integral a map Z which sends an embedded circle in $\mathbb{R}^3$ to an element in $\mathcal{A}(S^1)$ [14, 2]. It is naturally extended to a map sending embedded circles to an element in $\mathcal{A}(\coprod S^1)$, which is also denoted by Z. His main result is that Z is a link invariant, i.e., it is invariant under ambient isotopy of $\mathbb{R}^3$. So if we have a map W from $\mathcal{A}(\coprod S^1)$ to a ring R then $W \circ Z$ gives an R-valued link invariant. We call such a W a weight system.

3. The Conway potential function as Vassiliev invariants

In this section we will show that every coefficient of $\nabla_n(L)$ is a Vassiliev invariant.

Definition 3.1. For a Laurent series f with variables $h_1, h_2, \dots, h_n$, we denote by $c_{p_1,p_2,\dots,p_n}(f)$ the coefficient of $\prod_{i=1}^{n} {h_i}^{p_i}$ in f. We also denote by $C_p(f)$ the total degree p part of f, which is equal to $\sum_{\sum p_i = p} c_{p_1,p_2,\dots,p_n}(f) \prod_{i=1}^{n} {h_i}^{p_i}$.

Then we have the following lemma.

Lemma 3.2. *The coefficient $c_{p_1,p_2,\dots,p_n}(\nabla_n(L))$ is a Vassiliev invariant of type $\sum_{i=1}^{n} p_i + 1$. So $C_p(\nabla_n(L))$ is also a Vassiliev invariant of type $p+1$.*

Proof. Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be the link diagram as shown below where they are the same outside this figure. (This figure has already appeared in R. Hartley's paper [11, Proof of (4.2)].) We also let $\mathcal{L}^1$ be the singular link diagram which is the same as $\mathcal{L}_+$ and $\mathcal{L}_-$ outside the figure.

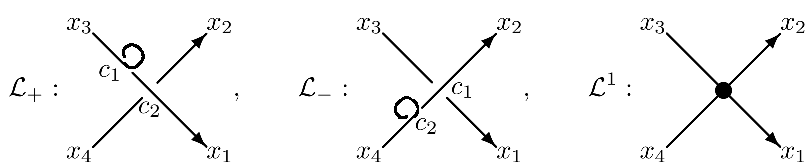


Figure 2

We assume that φ sends x_1 and x_3 to t_k and x_2 and x_4 to t_l (k and l may be the same). Then the Jacobians $M(\mathcal{L}_+)$ and $M(\mathcal{L}_-)$ are

$$M(\mathcal{L}_+) = \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & O \\ 1-t_l & t_k & 0 & -1 & O \\ \hline m_1 & m_2 & m_3 & m_4 & M \end{array} \right)$$

and

$$M(\mathcal{L}_-) = \left(\begin{array}{cccc|c} 1 & t_k-1 & -t_l & 0 & O \\ 0 & 1 & 0 & -1 & O \\ \hline m_1 & m_2 & m_3 & m_4 & M \end{array} \right)$$

for some column vectors m_1, m_2, m_3, m_4 and a matrix M, where O is a zero vector of suitable size. We choose $i, j > 4$ to calculate $D^{(ij)}(\mathcal{L}_\pm)$. Putting $\tilde{D}^{(ij)}(\mathcal{L}_\pm) = D^{(ij)}(\mathcal{L}_\pm) \prod_{g=1}^{n} {t_g}^{(\kappa_g(\mathcal{L}_\pm) - \nu_g(\mathcal{L}_\pm))/2}$ and $\tilde{D}^{(ij)}(\mathcal{L}^1) = \tilde{D}^{(ij)}(\mathcal{L}_+) - \tilde{D}^{(ij)}(\mathcal{L}_-)$, we have

$$\begin{aligned} \tilde{D}^{(ij)}(\mathcal{L}^1) = \prod_{g=1}^{n} {t_g}^{\varepsilon_g} \{ {t_k}^{-1/2} & \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & O \\ 1 & t_k & -t_l & -1 & O \\ \hline m'_1 & m'_2 & m'_3 & m'_4 & M' \end{array} \right\| \\ - {t_l}^{-1/2} & \left\| \begin{array}{cccc|c} 1 & t_k & -t_l & -1 & O \\ 0 & 1 & 0 & -1 & O \\ \hline m'_1 & m'_2 & m'_3 & m'_4 & M' \end{array} \right\| \} \\ = \prod_{g=1}^{n} {t_g}^{\varepsilon_g} \{ & \left\| \begin{array}{cccc|c} {t_k}^{-1/2} & 0 & -{t_k}^{-1/2} & 0 & O \\ 1 & t_k & -t_l & -1 & O \\ \hline m'_1 & m'_2 & m'_3 & m'_4 & M' \end{array} \right\| \end{aligned}$$

$$- \left\| \begin{array}{cccc|c} 1 & t_k & -t_l & -1 & O \\ 0 & {t_l}^{-1/2} & 0 & -{t_l}^{-1/2} & O \\ \hline m_1' & m_2' & m_3' & m_4' & M' \end{array} \right\| \}$$

$$= \prod_{g=1}^{n} {t_g}^{\varepsilon_g} \left\| \begin{array}{cccc|c} {t_k}^{-1/2} & {t_l}^{-1/2} & -{t_k}^{-1/2} & -{t_l}^{-1/2} & O \\ 1 & t_k & -t_l & -1 & O \\ \hline m_1' & m_2' & m_3' & m_4' & M' \end{array} \right\|$$

for some half integers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, some column vectors m_1', m_2', m_3', m_4', and some matrix M'. (Note that $\kappa_k(\mathcal{L}_+) = \kappa_k(\mathcal{L}_-) + 1, \kappa_l(\mathcal{L}_+) = \kappa_l(\mathcal{L}_-) - 1, \nu_k(\mathcal{L}_+) = \nu_k(\mathcal{L}_-) + 2, \nu_l(\mathcal{L}_+) = \nu_l(\mathcal{L}_-) - 2$.) Then the Conway potential function of the labelled singular link L^1 presented by $\mathcal{L}^1$ is given by

$$\nabla_n(L^1; t_1, t_2, \ldots, t_n) = \frac{(-1)^{i+j} \tilde{D}^{(ij)}(\mathcal{L}^1)}{\varphi(w_i)(\varphi(x_j) - 1)}.$$

Similarly we see that

$$\nabla_n(L^d; t_1, t_2, \ldots, t_n) = \frac{(-1)^{i+j} \tilde{D}^{(ij)}(\mathcal{L}^d)}{\varphi(w_i)(\varphi(x_j) - 1)}.$$

Here $\mathcal{L}^d$ is a singular link diagram with d double points presenting L^d and $\tilde{D}^{(ij)}(\mathcal{L}^d)$ is given as follows. We arrange $\mathcal{L}^d$ so that four arcs adjacent to each double points are different after inserting kinks if necessary. Then $\tilde{D}^{(ij)}(\mathcal{L}^d)$ is of the form

$$\prod_{g=1}^{n} {t_g}^{\varepsilon_g} \left\| \begin{array}{cccccc|c} T_1 & O & O & \cdots & O & O \\ * & T_2 & O & \cdots & O & O \\ * & * & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O & O \\ * & * & \cdots & * & T_d & O \\ \hline * & * & \cdots & * & * & * \end{array} \right\|,$$

where T_e is a 2×4 matrix of the form

$$\begin{pmatrix} {t_{k(e)}}^{-1/2} & {t_{l(e)}}^{-1/2} & -{t_{k(e)}}^{-1/2} & -{t_{l(e)}}^{-1/2} \\ 1 & t_{k(e)} & -t_{l(e)} & -1 \end{pmatrix}.$$

(The author does not know whether the matrix above can be derived from Fox's free differential calculus applied to $\mathcal{L}^d$.) It is not hard to see that the total degree of $\tilde{D}^{(ij)}(\mathcal{L}^d)$ is at least d putting $t_k = \exp(h_k)$. Therefore the total degree of $\nabla_n(L^d)$ is at least $d-1$. This shows that $c_{p_1,p_2,\ldots,p_n}(\nabla_n(L^d))$ vanishes if $\sum_{k=1}^n p_k < d-1$ and so $c_{p_1,p_2,\ldots,p_n}(\nabla_n(L))$ is a Vassiliev invariant of type $\sum_{k=1}^n p_k + 1$, completing the proof. □

4. A weight system

In this section we use J. Murakami's relations [19, p. 126, (1)–(6)] to define a weight system W_n.

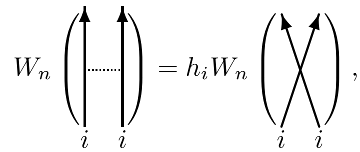
Definition 4.1. For a chord diagram D with d chords, we put $W_n(D) = C_{d-1}(\nabla_n(D))$ and extend it linearly to a map from $\mathcal{D}(\coprod^{\mu} S^1)$.

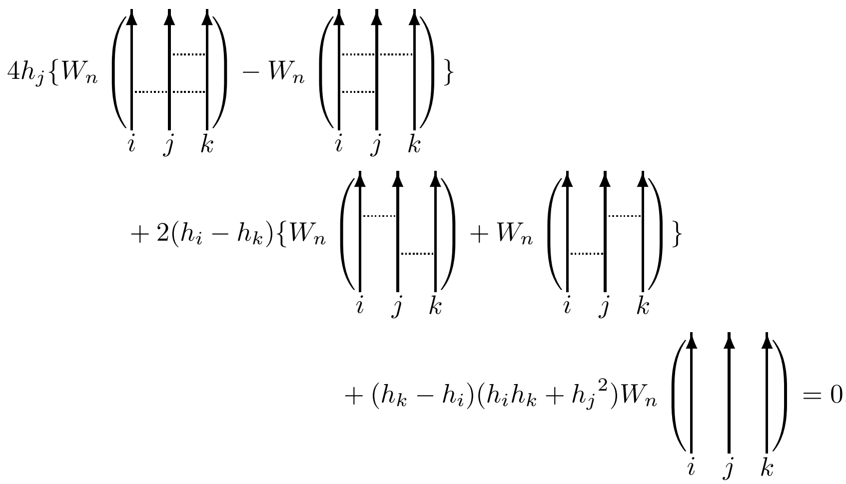
Since C_{d-1} is a Vassiliev invariant of type d and D has d chords $W_n(D)$ does not depend on its embedding and so is well defined as a map from $\mathcal{A}(\coprod^{\mu} S^1)$ to the set

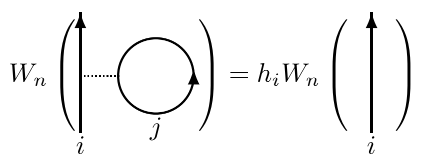
of homogeneous polynomials in $h_1, h_2, \ldots, h_n$ (if $\mu = 1$ it also contains a term of the form $c{h_1}^{-1}$). For a proof that W_n satisfies the 4-term relation and the framing independent relation, see for example [5, 4].

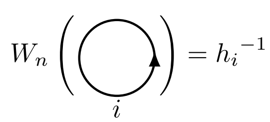
Now we will characterise W_n.

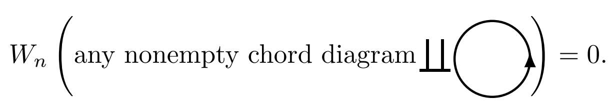
Proposition 4.2. *W_n satisfies the following formulae.*

(4.1)


(4.2)


(4.3)


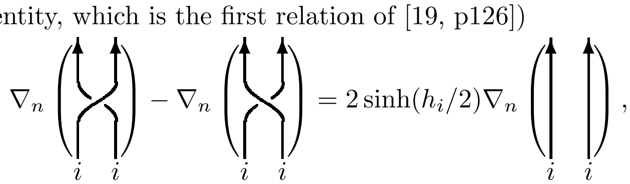
(4.4)


(4.5)


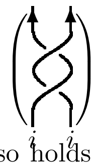
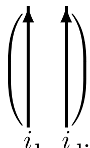
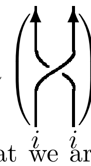
Here i, j, k indicate the labels attached to the arcs near by. Note that some of the labels i, j, k may be equal. Note also that the crossing in the right hand side of (4.1) is not *a double point. It only indicates the connectivity.*

Proof. We assume that the chord diagrams appearing in the lemma have d double points outside the regions described in the pictures.

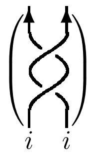
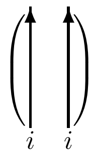
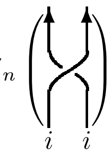
(Proof of (4.1)) From the well-known relation for the potential function (Conway's first identity, which is the first relation of [19, p126])

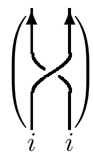


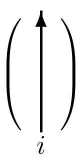
we have

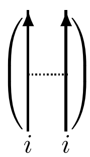
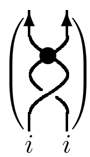
∇_n  $-\ \nabla_n$  $= 2\sinh(h_i/2)\nabla_n$  .

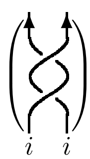
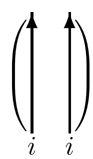
Note that this also holds for singular links and recall that we are assuming that there are d double points outside the region appearing in the equality above. Taking the total degree d part, we have

(4.6) $C_d(\nabla_n$  $) - C_d(\nabla_n$  $) = 2C_1(\sinh(h_i/2))C_{d-1}(\nabla_n$  $)$

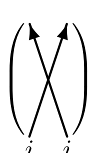
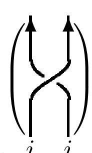
$= h_i C_{d-1}(\nabla_n$  $)$

since $C_e(\nabla_n$  vanishes if $e < d-1$ from Lemma 3.2. By the way, we have from the definition

W_n  $= C_d(\nabla_n$  $)$

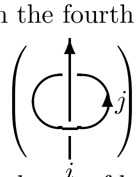
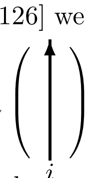
$= C_d(\nabla_n$  $) - C_d(\nabla_n$  $)$

and

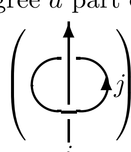
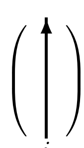
W_n  $= C_{d-1}(\nabla_n$  $)$.

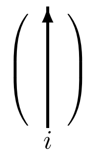
Therefore the required formula follows from (4.6).

(Proof of (4.3)) From the fourth relation of [19, p.126] we have

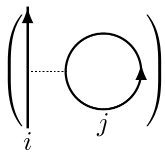
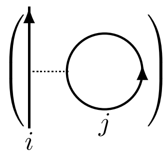
∇_n  $= 2\sinh(h_i/2)\nabla_n$  .

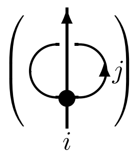
Taking the total degree d part of both hand sides, we have

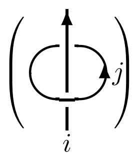
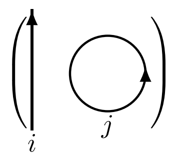
(4.7) $C_d(\nabla_n$  $) = 2C_1(\sinh(h_i/2))C_{d-1}(\nabla_n$  $)$

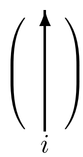
$$= h_i C_{d-1}(\nabla_n$$  $$).$$

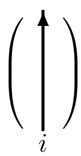
So we have

$$W_n$$  $$= C_d(\nabla_n$$  $$)$$

$$= C_d(\nabla_n$$  $$)$$

$$= C_d(\nabla_n$$  $$) - C_d(\nabla_n$$  $$)$$

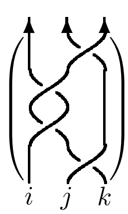
$$= h_i C_{d-1}(\nabla_n$$  $$)$$

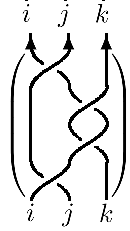
$$= h_i W_n$$  $$.$$

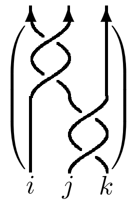
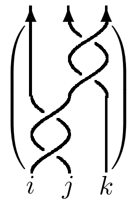
Here we use (4.7) and the fact that ∇_n vanishes for a split link in the fourth equality.

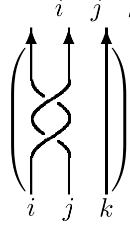
(Proof of (4.4) and (4.5)) Since $\nabla_n(O_i) = 1/(2\sinh(h_i/2))$ (which is the fifth relation of [19, p. 126]), $W_n(O_i) = C_{-1}(1/(2\sinh(h_i/2))) = {h_i}^{-1}$ and we have (4.4). Here O_i is the trivial knot with label i. The relation (4.5) follows from the fact that ∇_n vanishes for split links (the sixth relation of [19, p. 126]).

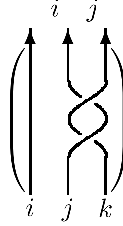
(Proof of (4.2)) We use J. Murakami's third relation [19, p.126]:

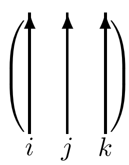
$$4\cosh(h_i/2)\sinh(h_j/2)\nabla_n$$ 

$$-4\cosh(h_k/2)\sinh(h_j/2)\nabla_n$$ 

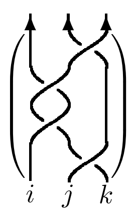
$-2\sinh((-h_i + h_k)/2)\{\nabla_n$  $+ \nabla_n$  $\}$

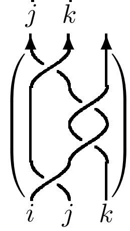
$+4\cosh(h_k/2)\sinh((-h_i + h_j + h_k)/2)\nabla_n$ 

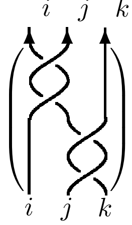
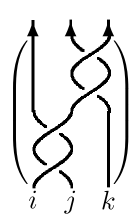
$-4\cosh(h_i/2)\sinh((h_i + h_j - h_k)/2)\nabla_n$ 

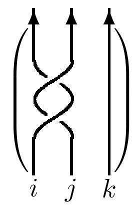
$-2\sinh(-h_i + h_k)\nabla_n$  $= 0.$

We take the total degree $d+2$ part. (Recall that we assume that there are d double points outside.) Since $C_e(L^d) = 0$ for $e < d-1$ and any singular link L^d with d double points, we have

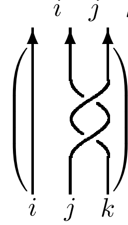
$4C_1(\cosh(h_i/2)\sinh(h_j/2))C_{d+1}(\nabla_n$  $)$

$-\, 4C_1(\cosh(h_k/2)\sinh(h_j/2))C_{d+1}(\nabla_n$  $)$

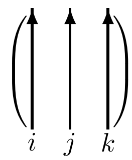
$-\, 2C_1(\sinh((-h_i + h_k)/2))\{C_{d+1}(\nabla_n$  $) + C_{d+1}(\nabla_n$  $)\}$

$$+\,4C_1(\cosh(h_k/2)\sinh((-h_i+h_j+h_k)/2))C_{d+1}(\nabla_n$$ 

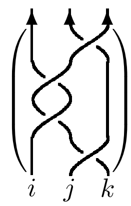
$)$

$$-\,4C_1(\cosh(h_i/2)\sinh((h_i+h_j-h_k)/2))C_{d+1}(\nabla_n$$ 

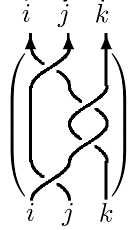
$)$

$$-\,2C_1(\sinh(-h_i+h_k))C_{d+1}(\nabla_n$$ 

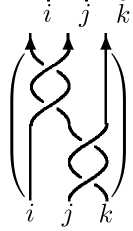
$)$

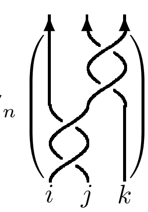
$$+\,4C_3(\cosh(h_i/2)\sinh(h_j/2))C_{d-1}(\nabla_n$$ 

$)$

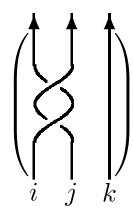
$$-\,4C_3(\cosh(h_k/2)\sinh(h_j/2))C_{d-1}(\nabla_n$$ 

$)$

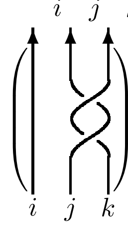
$$-\,2C_3(\sinh((-h_i+h_k)/2))\{C_{d-1}(\nabla_n$$ 

$) + C_{d-1}(\nabla_n$ 

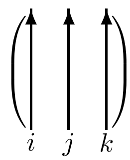
$)\}$

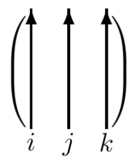
$$+\,4C_3(\cosh(h_k/2)\sinh((-h_i+h_j+h_k)/2))C_{d-1}(\nabla_n$$ 

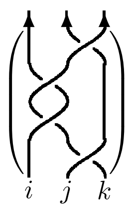
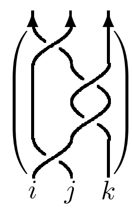
$)$

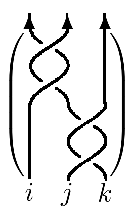
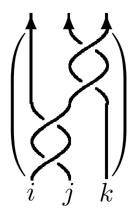
$$-\,4C_3(\cosh(h_i/2)\sinh((h_i+h_j-h_k)/2))C_{d-1}(\nabla_n$$ 

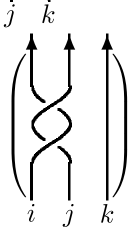
$)$

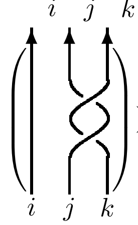
$- 2C_3(\sinh(-h_i + h_k))C_{d-1}(\nabla_n$  $) = 0.$

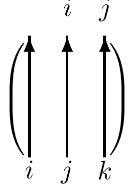
Now since C_{d-1} is a type d invariant, all its values in the equality above are the same and equal to $C_{d-1}(\nabla_n$  $)$. So we have

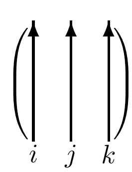
(4.8) $2h_j\{C_{d+1}(\nabla_n$  $) - C_{d+1}(\nabla_n$  $)\}$

$- (-h_i + h_k)\{C_{d+1}(\nabla_n$  $) + C_{d+1}(\nabla_n$  $)\}$

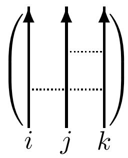
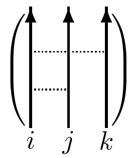
$+ 2(-h_i + h_j + h_k)C_{d+1}(\nabla_n$  $)$

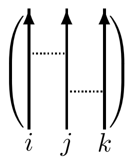
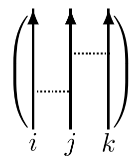
$- 2(h_i + h_j - h_k)C_{d+1}(\nabla_n$  $)$

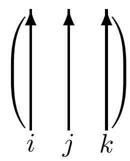
$- 2(-h_i + h_k)C_{d+1}(\nabla_n$  $)$

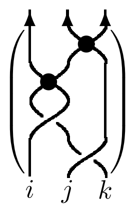
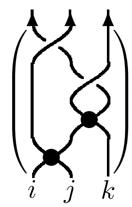
$+ \frac{1}{2}(h_k - h_i)(h_ih_k + h_j^2)C_{d-1}(\nabla_n$  $) = 0.$

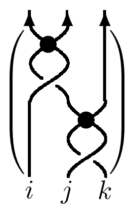
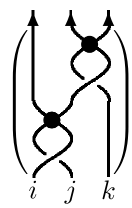
Now we have from the definition of W_n

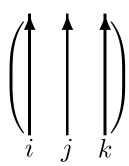
$2h_j\{W_n$  $- W_n$  $\}$

$+ (h_i - h_k)\{W_n$  $+ W_n$  $\}$

$+ \frac{1}{2}(h_k - h_i)(h_i h_k + {h_j}^2)W_n$ 

$=2h_j\{C_{d+1}(\nabla_n$  $) - C_{d+1}(\nabla_n$  $)\}$

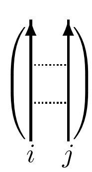
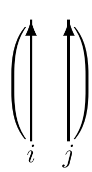
$+ (h_i - h_k)\{C_{d+1}(\nabla_n$  $) + C_{d+1}(\nabla_n$  $)\}$

$+ \frac{1}{2}(h_k - h_i)(h_i h_k + {h_j}^2)C_{d+1}(\nabla_n$  $),$

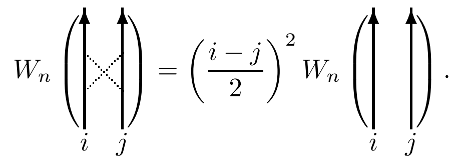
which vanishes from (4.8) and the proof is complete. □

Remark 4.3. In the proof above we did not use Conway's second identity (the second relation of [19, p. 126]. The author does not know whether it is necessary in J. Murakami's axioms for the multivariable Alexander polynomial. In our case we have the following corollary which corresponds to Conway's second identity.

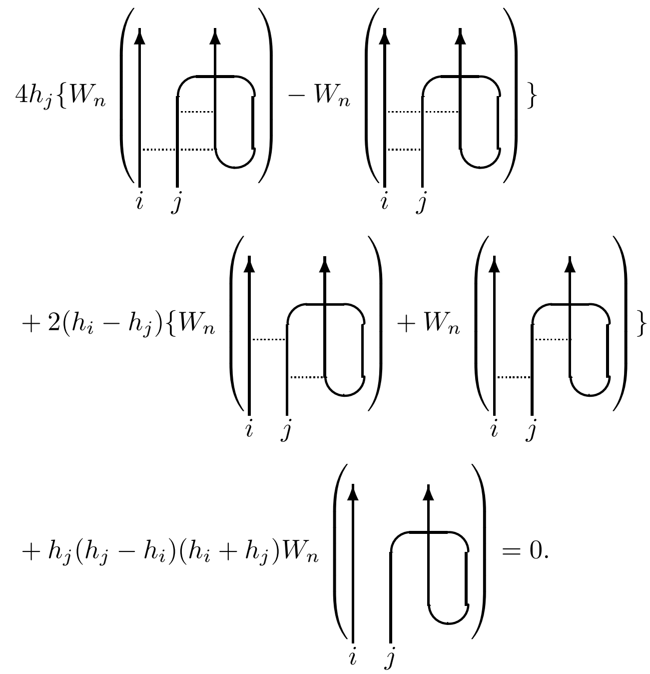
Corollary 4.4. *We have*

W_n  $= \left(\frac{i+j}{2}\right)^2 W_n$ 

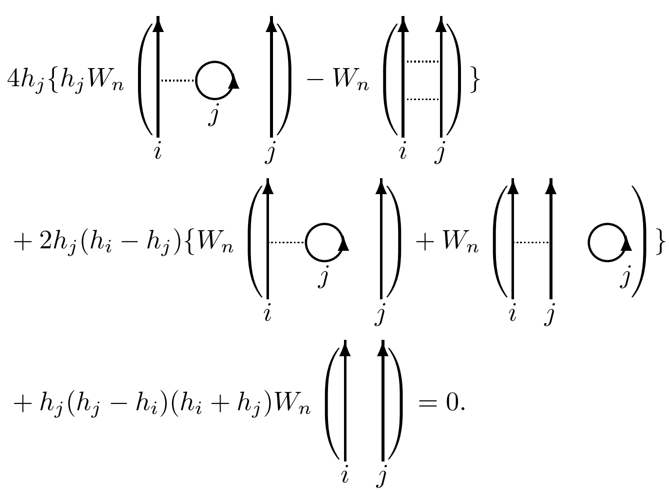
and



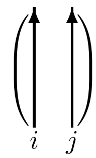
Proof. We put $k = j$ in the relation (4.2) and connect these two arcs as follows.

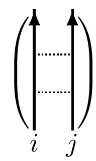


Then applying the relation (4.1), we have



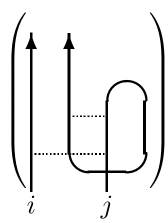
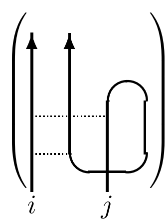
Now using the relations (4.3) and (4.5), we have

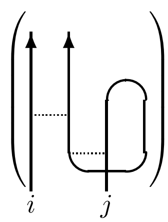
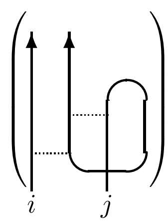
$$\{4h_j^2 h_i + h_j(h_j - h_i)(h_i + h_j) + 2h_i h_j(h_i - h_j)\}W_n$$


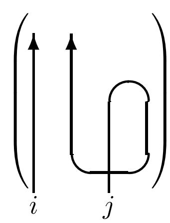
$$- 4h_j W_n$$

$$= 0.$$

So the required formula follows.

Similarly the following connection shows the second formula.

$$4h_j\{W_n$$

$$- W_n$$

$$\}$$

$$+ 2(h_i - h_j)\{W_n$$

$$+ W_n$$

$$\}$$

$$+ h_j(h_j - h_i)(h_i + h_j)W_n$$

$$= 0.$$

□

J. Murakami proved that his six relations are sufficient to calculate the multivariable Alexander polynomial. Now our main result is

Theorem 4.5. *W_n can be calculated recursively by using axioms (4.1) – (4.5).*

Proof. We proceed by induction on the number of circles in the support of a chord diagram. If there is only one circle, then we use (4.1) to change the chord diagram into a diagram without chords. Then we apply (4.4) or (4.5) to evaluate the diagram.

Suppose that we are given a chord diagram D with support $E_1, E_2, \dots, E_\mu$ ($\mu > 1$). We first look at E_1. If E_1 contains no end point of a chord, then $W_n(D) = 0$ from (4.5). If E_1 contains one end point, then from (4.3) we can reduce the number of circles. If E_1 contains more than one end point, we use (4.2). We assume that E_1 is labelled i in (4.2). The second term there contains two end points and the others

contain one or less. So we can reduce the number of end points. Repeating this process we have chord diagrams with one or no end point, which can be calculated as described above.

So the proof is complete. □

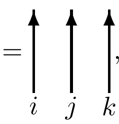
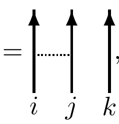
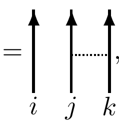
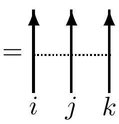
5. Problems and a conjecture

In this section we discuss open problems and state a conjecture. Our first problem is

Problem 5.1. Show that W_n is well-defined without using the Conway potential function.

Note that the relation (4.2) implies the 4-term relation as follows. We can write (4.2) as

$$4h_j(BC - CA) + 2(h_i - h_k)(AB + BA) + (h_k - h_i)(h_i h_k + {h_j}^2)I = 0$$

with $I =$ , $A =$ , $B =$ , and $C =$  and we composite them downward. If we exchange the labels i and k, then A and B are also exchanged and so we have

$$4h_j(AC - CB) + 2(h_k - h_i)(AB + BA) + (h_i - h_k)(h_i h_k + {h_j}^2)I = 0.$$

Adding the two equalities above and divide by $4h_j$, we have $AC - CA = CB - BC$, which is the 4-term relation.

It is easily seen that (4.1) and (4.5) imply the framing independence relation. Therefore the well-definedness of W_n as a map from $\mathcal{D}(\coprod S^1)$ implies that it factors through $\mathcal{A}(\coprod S^1)$, which proves that W_n defines a link invariant via the Kontsevich integral!

The next problem is

Problem 5.2. Can we alter the coefficients appeared in Proposition 4.2?

For example, let us replace (4.2) with

$$x(h_i, h_j, h_k)(BC - CA) + y(h_i, h_j, h_k)(AB + BA) + z(h_i, h_j, h_k)I = 0, \tag{5.1}$$

where $x(h_i, h_j, h_k), y(h_i, h_j, h_k)$ and $z(h_i, h_j, h_k)$ are functions of h_i, h_j, and h_k. If $x(h_i, h_j, h_k)$ is (nonzero and) symmetric with respect to h_i and h_k, and $y(h_i, h_j, h_k)$ and $z(h_i, h_j, h_k)$ are antisymmetric with respect to h_i and h_k, then the relation (5.1) above implies the 4-term relation. So if we could prove that $\mathcal{D}(\coprod S^1)$ modulo (5.1), (4.1), (4.3), (4.4), and (4.5) is nontrivial, we might have another labelled link invariant. Our W_n is a map to homogeneous polynomials but the author does not know whether the homogeneity is necessary or not.

As D. Bar-Natan and S. Garoufalidis pointed out in [3], W_1 is a canonical weight system in the sense that $W_1 \circ Z$ coincides with ∇_1. (Note that their definition of the weight system is $h_1 W_1$ in our notation.) Our conjecture is

Conjecture 5.3. *W_n is canonical for every n, i.e., $W_n \circ Z = \nabla_n$.*

To prove this it is sufficient to prove that $W_n \circ Z$ satisfies the six axioms in [19, p. 126]. The author cannot prove it since they involve Drinfel'd's associator

[7, 8, 17, 16, 1, 12]. Note that there is no direct proof (using Drinfel'd's associator) of the equality $W_1 \circ Z(O_1) = 1/(2\sinh(h_1))$. The proof in [3, Example 2.7] depends on T.Q.T. Le and J. Murakami's result on the canonical weight system for the HOMFLY polynomial [15] which depends on the skein relation not on Drinfel'd's associator (in fact they use the skein relation to prove some interesting formulae about coefficients of Drinfel'd's associator which involve the multiple zeta functions). Note also that since $W_1 = W_n\big|_{h_1=h_2=\cdots=h_n}$, we have $(W_n \circ Z)\big|_{h_1=h_2=\cdots=h_n} = \nabla_n\big|_{h_1=h_2=\cdots=h_n}$.

References

[1] D. Bar-Natan, *Non-associative tangles*, to appear in the Georgia International Topology Conference proceedings.

[2] ———, *On the Vassiliev knot invariant*, Topology **34** (1995), 423–472.

[3] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. **125** (1996), 103–133.

[4] J.S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), 253–287.

[5] J.S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[6] J.H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (J. Leech, ed.), Pergamon, 1970, pp. 329–358.

[7] V.G. Drinfel'd, *On quasi-Hopf algebras*, Leningrad Math. J. **1** (1990), 1419–1457.

[8] ———, *On quasi-triangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Leningrad Math. J. **2** (1990), 829–860.

[9] R.H. Fox, *Free differential calculus. I*, Ann. of Math. (2) **57** (1953), 547–560.

[10] ———, *Free differential calculus. II*, Ann. of Math. (2) **59** (1954), 196–210.

[11] R. Hartley, *The Conway potential function for links*, Comment. Math. Helv. **58** (1983), 365–378.

[12] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, 1995.

[13] M.E. Kidwell, *On the two-variable Conway potential function*, Proc. Amer. Math. Soc. **98** (1986), 485–494.

[14] M. Kontsevich, *Vassiliev's knot invariants*, Advances in Soviet Mathematics **16** (1993), 137–150.

[15] T.Q.T. Le and J. Murakami, *Kontsevich's integral for the HOMFLY polynomial and relations between values of multiple zeta functions*, Topology Appl. **62** (1995), 193–206.

[16] ———, *Representations of the category of tangles by Kontsevich's iterated integral*, Comm. Math. Phys. **168** (1995), 535–562.

[17] ———, *Kontsevich integral for the Kauffman polynomial*, Nagoya Math. J. **142** (1996), 39–65.

[18] P.M. Melvin and H.R. Morton, *The coloured Jones function*, Comm. Math. Phys. **169** (1995), 501–520.

[19] J. Murakami, *A state model for the multi-variable Alexander polynomial*, Pacific J. Math. **157** (1993), 109–135.

[20] Y. Nakanishi, *Three-variable Conway potential function of links*, Tokyo J. Math. **13** (1990), 163–177.

[21] L. Rozansky, *A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175** (1996), 275–296.

[22] V.A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and Its Applications (V.I. Arnold, ed.), Advances in Soviet Math., vol. 1, Amer. Math. Soc., Providence, 1990.

Department of Mathematics, Osaka City University, 3-138, Sugimoto 3-chome, Sumiyoshi-ku, OSAKA 558, JAPAN, and, Department of Pure Mathematics, University of Liverpool, PO Box 147, LIVERPOOL L69 3BX, UK

Current address: Department of Mathematics, School of Science and Engineering, Waseda University, 4-1, Ohkubo 3-chome, Shinjuku-ku, TOKYO 169, JAPAN